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COMMENT

Comment on quantum representations of non-bijective canonical transformations

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Abstract. A recent letter by J Deenen on 'non-bijective canonical transformations in quantum mechanics' is analysed and found to be incomplete. The general representation problem of non-bijective canonical transformations is shown to involve infinite sums of the quotient and tensor product spaces defined by Plebański and Seligman.

In a recent letter Deenen (1981) proposed an embedding of the quantum mechanical Hilbert space $\mathscr{H}$ into a larger space in order to be able to represent all canonical transformations on a single space given by

$$\mathbb{H} = \mathscr{H} \times \mathscr{H}. \tag{1}$$

He then proposes a set of semi-unitary operators U_σ; $\sigma = 1, \ldots, k$ with the properties

$$\begin{aligned} &U_\sigma U_\sigma^+ = \mathbb{1} \\ &U_\sigma^+ U_\sigma = P_\sigma \\ &\sum_{\sigma=1}^{k} P_\sigma = \mathbb{1}, \qquad P_\sigma P_{\sigma'} = \delta_{\sigma,\sigma'} P_\sigma \end{aligned} \tag{2}$$

where $\mathbb{1}$ is the identity and P_σ a projection operator. Then

$$\mathbb{U} = \sum_\sigma U_\sigma \otimes U_\sigma^+ \tag{3}$$

is clearly a unitary operator acting on $\mathbb{H}$ if the first and second terms are considered to act on the respective factors of $\mathbb{H}$. This scheme is readily generalisable to more general partial isometries as discussed by Plebański and Seligman (1982) and thus it seems quite satisfactory in eliminating the complicated transformation-dependent structures given in this reference.

Unfortunately this mapping has no representation properties even for a linear canonical transformation. The usual representation $U(s)$ is a homomorphic ray representation (Kramer *et al* 1975)

$$U(s)U(s') = \exp(\mathrm{i}\varphi(s,s', ss'))U(ss') \tag{4a}$$

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or possibly an antihomomorphic one (Moshinsky and Quesne 1971).

$$U(s)U(s') = \exp(\mathrm{i}\varphi(s, s', s's))U(s's). \qquad (4b)$$

For the operator

$$\mathbb{U}(s) = U(s) \otimes U^{+}(s) \qquad (5)$$

we get

$$\mathbb{U}(s)\mathbb{U}(s') = \exp[\mathrm{i}(\varphi(s, s', s's) - \varphi(s, s', ss'))]U(ss') \otimes U^{+}(s's). \qquad (6)$$

Therefore we obtain a mixture of a homomorphism and an antihomomorphism for the two spaces. Thus even the group of linear canonical transformations is not represented in this way and successive applications of non-bijective canonical transformations lead to similar complications. In consequence the method proposed by Deenen (1981) is inadequate.

On the other hand multiplication of $\mathscr{H}$ with some separable Hilbert space $\mathscr{H}'$ yields an appropriate basis to represent a wide class of canonical transformations.

Take for instance all transformations between oscillators of different frequencies

$$p^2 + q^2 = \frac{1}{\alpha}(\bar{p}^2 + \bar{q}^2), \qquad \tan^{-1} p/q = \alpha \tan^{-1} \bar{p}/\bar{q}. \qquad (7)$$

Clearly successive applications lead locally to a one parameter group corresponding to the multiplication of positive real numbers. In terms of action and angle variables these are simple dilations as pointed out by Deenen and Moshinsky (1981). They thus form a group of linear transformations $Q = (1/\alpha)\bar{Q}, P = \alpha\bar{P}$. The transformation to these variables may be represented by choosing $\mathscr{H}' = \mathscr{L}^2(\mathbb{Z})$, $\mathbb{Z}$ being the unit circle (Moshinsky and Seligman 1978, 1981) and thus any transformation of the type given in (7) is homomorphically represented on

$$\mathscr{H} \times \mathscr{L}^2(\mathbb{Z}). \qquad (8)$$

On the other hand successive applications of transformations to action and angle variables *cannot* be represented in this space but require successive enlargements or quotient formation for the inverse as defined in equations (5.6) of Plebański and Seligman.

A way to define all operations as acting on one space is to form a 'Fock space'

$$\mathbb{H} = \cdots \oplus \mathscr{H}/\mathscr{C}_\infty/\mathscr{C}_\infty \oplus \mathscr{H}/\mathscr{C}_\infty \oplus \mathscr{H} \oplus \mathscr{H} \times \mathscr{L}^2(\mathbb{Z}) \oplus \cdots \qquad (9)$$

as an infinite sum of quotient and tensor product spaces as defined in Plebański and Seligman. $\mathbb{H}$ is still a separable Hilbert space.

This is consistent with the fact that the product of two successive transformations to action and angle variables has the ambiguity group

$$(T \wedge J) \times (T \wedge J) \qquad (10)$$

where T indicates a discrete translation and J an inversion group.

Explicitly the product of two successive transformations reads as

$$\begin{aligned} \bar{\bar{q}} &= (2\sqrt{2|q|}\,|\cos p|)^{1/2} \cos\left(\sqrt{2|q|} \sin \frac{pq}{|q|}\right) \\ \bar{\bar{p}} &= (2\sqrt{2|q|}\,|\cos p|)^{1/2} \sin\left(\sqrt{2|q|} \sin \frac{pq}{|q|} \frac{\cos p}{|\cos p|}\right). \end{aligned} \qquad (11)$$

The ambiguity group is generated by the operations

$$
\begin{aligned}
&q \to -q, \qquad p \to -p \\
&q \to q, \qquad p \to p + 2m\pi; \qquad && m \text{ integer} \\
&\sqrt{2|q|}\sin(pq/|q|) \to -\sqrt{2|q|}\sin(pq/|q|), \qquad && \cos p \to -\cos p \qquad (12) \\
&\sqrt{2|q|}\sin(pq/|q|) \to \sqrt{2|q|}\sin(pq/|q|) + 2m\pi; \qquad && m \text{ integer.}
\end{aligned}
$$

A careful analysis shows that the first pair of operations commutes with the second pair and thus we have the group indicated above.

We may now enquire whether the two methods mentioned above to represent canonical transformations presented are useful.

The first method mentioned involved finding one non-bijective canonical transformation that will take a family of non-bijective canonical transformations into a family of bijective ones. If the original family is a 'local group' the image will be a group. Such a procedure is useful but of limited domain as there remains at least the first transformation.

The second method of introducing both expanded and reduced spaces in a Fock-type sum is formally adequate and general. It seems to be the correct answer to the question posed by Deenen (1981) but it is complicated as it involves ever more sums, because we may need additional tensor products and quotients for other types of transformations. Indeed we have no proof that a denumerable set of terms in the sum will cover all canonical transformations. If not this would lead to non-separable spaces.

We thus conclude that while we have found a correct answer to the question posed by Deenen, the very structure of this answer implies that the attempt to use a Hilbert space large enough to carry a unitary representation of all canonical transformations tends to complicate rather than to simplify the problem in hand.

Finally, note (Moshinsky and Seligman 1981) that this problem is structurally identical with the one encountered if we attempt to obtain a global group of canonical transformations from the local one or a global group of conformal mappings from the local one. Thus for these problems it again seems desirable to introduce Riemann sheets or ambiguity spins for each transformation individually or for small well defined families of transformations only, and to dismiss the problem of searching for a global group.

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